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EFFECT OF THE COUNTERPRESSURE IN THE ONE-DIMENSIONAL UNSTEADY FLOW OF A VISCOUS GAS

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One of the possible approaches in investigating one-dimensional flows of a viscous heat conducting gas, connected with the assumption that counterpressure is absent, and this in turn implies the existence of a surface of weak discontinuity, i.e. a perturbation front [1]. The perturbation front appears only when the speed of sound in the unperturbed region is strictly equal to zero. It will be shown below that if this speed of sound is not zero but is sufficiently small compared with the mean velocity of propagation of the perturbations, then the asymptotic analysis makes it possible to replace the weak discontinuity by a certain structured region of smooth variation in the hydrodynamic parameters, which is equivalent to accounting for the effect of the counterpressure. The order of the thickness of the above transitional region is estimated, and the problem of flow within this region is reduced to a single quadrature which can be computed, in some cases, using the analytic methods.

Let us consider a one-dimensional unsteady flow of a viscous heat conducting gas. The gas is assumed to be perfect, with Prandtl number σ, ratio of specific heats $\varkappa$ and power index n in the law showing the independence of viscosity on temperature, all constant. We first assume that a region of unperturbed state exists (with the corresponding parameters denoted by the subscript 1) in which the gas is at rest when the temperature and pressure are both zero. Then the region of perturbed motion will be bounded by the perturbation front $r = r_f(t)$ moving with a finite velocity $U = dr_f / dt$.

Using a standard notation for the dimensional hydrodynamic parameters, we can write the Navier-Stokes equations in the dimensionless form (see [1]) replacing r and t by the new arguments

$$\eta = \frac{r}{r_f}, \quad \chi = \frac{\mu_0 U^{2n-1}}{\rho_1 [(\varkappa - 1) c_v T_0^{\cdot}]^n r_f}$$

where the zero subscript refers to some standard state, used here as the initial state.

We reduce the unknown quantities to the dimensionless form using the formulas

$$v = UV(\eta, \chi), \quad \rho = \rho_1 R(\eta, \chi), \quad p = \rho_1 U^2 P(\eta, \chi)$$

$$T = c_v^{-1} (\varkappa - 1)^{-1} U^2 N(\eta, \chi), \quad \mu = \chi \rho_1 r_f U N^n(\eta, \chi)$$

In the new variables the Navier-Stokes equations become

$$(V - \eta)\frac{\partial R}{\partial \eta} + K\chi \frac{\partial R}{\partial \chi} + R\frac{\partial V}{\partial \eta} + \frac{\nu - 1}{\eta} RV = 0 \tag{1}$$

$$R\left[ZV + (V - \eta)\frac{\partial V}{\partial \eta} + K\chi\frac{\partial V}{\partial \chi}\right] + \frac{\partial P}{\partial \eta} =$$

$$\frac{4}{3}\chi\frac{\partial}{\partial \eta}\left[N^n\left(\frac{\partial V}{\partial \eta} - \frac{\nu - 1}{2}\frac{V}{\eta}\right)\right] + 2(\nu - 1)\chi\frac{N^n}{\eta}\left(\frac{\partial V}{\partial \eta} - \frac{V}{\eta}\right)$$

$$R\left[2ZN + (V-\eta)\frac{\partial N}{\partial \eta} + K\chi\frac{\partial N}{\partial \chi} + (\varkappa - 1)N\left(\frac{\partial V}{\partial \eta} + \frac{\nu - 1}{\eta}V\right)\right] =$$
$$\frac{\varkappa}{\sigma}\chi\eta^{1-\nu}\frac{\partial}{\partial \eta}\left(\eta^{\nu-1}N^n\frac{\partial N}{\partial \eta}\right) + 2(\varkappa - 1)\chi N^n\left\{\left(\frac{\partial V}{\partial \eta}\right)^2 + (\nu - 1)\frac{V^2}{\eta^2} -\right.$$
$$\left.\frac{1}{3}\left[\frac{\partial V}{\partial \eta} + (\nu - 1)\frac{V}{\eta}\right]^2\right\}, \quad P = RN$$

$$Z(\chi) = r_f\,(dU/dt)/U^2, \quad K(\chi) = (2n - 1)Z(\chi) - 1$$

where the parameter ν characterizes the type of symmetry ($\nu = 1, 2, 3$).

If we remove the initial assumption of the absence of counterpressure, then the perturbations will, in general, propagate without limit and the concept of a perturbation front will lose its initial sense. In many cases however, the functions $r_f(t)$ and $U(t)$ can still be used and intepreted as the mean distance and mean velocity of propagation of the perturbations.

Let the counterpressure p_1 be not zero, although much smaller than the quantity $\rho_1 U^2$, so that $P_1 = \varepsilon\ (\varepsilon \ll 1)$. The quantity ε can be regarded as the ratio of the square of the speed of sound in the unperturbed gas to the square of the mean velocity of propagation of perturbations. The ratio is assumed here to be small.

Let us now use the asymptotic method and consider the neighborhood of the surface $\eta = 1$ as a region with a specified order of smallness in ε. The solution constructed for this region is matched asymptotically with the known solution for the outer region of motion where $\varepsilon = 0$ and $\eta \to 1$ on one side, and with the solution for the outer region at rest where $P \to \varepsilon$, on the other side.

Let us perform the change of variables

$$P = \varepsilon P_*, \quad N = \varepsilon N_*, \quad V = \varepsilon^\alpha V_*, \quad \eta = 1 - \varepsilon^\beta z_*^\circ \tag{2}$$

and choose the constant indices α and β using the conditions of matching with the outer region of the flow. When $z_*^\circ \to \infty$, the solution expressed in terms of the functions accompanied by an asterisk should match with the limiting form of the solution obtained without accounting for the counterpressure for $1 - \eta \to 0$. We use for this region the asymptotic formulas suitable for the case $\zeta = 3\varkappa / (4\sigma) > 1$ (see [2])

$$N = P = A_N(1-\eta)^{1/n}, \quad V = \zeta(\zeta - 1)^{-1}A_N(1-\eta)^{1/n} \tag{3}$$
$$A_N = [\sigma n(\varkappa\chi)^{-1}]^{1/n}$$

The conditions of matching yield, together with (3), $\alpha = 1$, $\beta = n$. Subsequent substitution of (2) into (1) yields, after a passage to the limit $\varepsilon \to 0$, simpler equations which contain the argument χ only as the parameter.

To exclude χ from our discussion completely, we perform another change of argument

$$z_* = z_*^\circ \sigma n / (\varkappa\chi)$$

after which (5) yields the following system of equations:

$$\frac{dR_*}{dz_*} = 0, \quad nN_*^n\frac{dN_*}{dz_*} = N_* - 1 \tag{4}$$
$$\frac{1}{\zeta}nN_*^n\frac{dV_*}{dz_*} = V_* - N_* + 1, \quad P_* = R_*N_*$$

in which the second and third equations are given in the form obtained by integrating once and taking into account the boundary conditions with $z_* \to -\infty$.

The complete set of boundary conditions for the initial system of equations has the following form in the new variables:

$$R_*(-\infty) = P_*(-\infty) = N_*(-\infty) = 1, \quad V_*(-\infty) = 0 \tag{5}$$

$$N_* \to z_*^{1/n}, \quad P_* \to z_*^{1/n}, \quad V_* \to \zeta(\zeta - 1)^{-1} z_*^{1/n} \quad \text{as} \quad z_* \to \infty$$

Formally, on passing to Eqs. (4) we need to retain, out of the conditions (5), only the following three: those for $R_*(-\infty)$, $N_*(\infty)$ and for $V_*(\infty)$.

It can easily be shown that the unknown R_*, P_* and V_* can be excluded from our considerations by assuming

$$R_* \equiv 1, \quad P_* = N_*, \quad V_* = \zeta(\zeta - 1)^{-1}(N_* - 1)$$

Thus we can reduce the problem to that of solving a unique first order equation for N_* which appears in (4). Replacing $N_*^n = u$, we can write this equation in the form

$$\int \frac{u^{1/n} du}{u^{1/n} - 1} = z_* + C \tag{6}$$

where the constant C is determined with due regard to conditions (5). The integral in (6) can be expressed in terms of the elementary functions if $1 / n = k$ where k is an integer. Restricting ourselves to the range $1/2 \leqslant n \leqslant 1$ we find that the analytic representation of the solution of the problem of the structure of the perturbation front is possible for the end points of this interval, namely

$$z_* = \begin{cases} \sqrt{N_*} + \frac{1}{2} \ln \frac{\sqrt{N_*} - 1}{\sqrt{N_*} + 1}, & n = \frac{1}{2} \\ N_* + \ln(N_* - 1), & n = 1 \end{cases} \tag{7}$$

Numerical quadrature is used to determine the relationship $N_*(z_*)$ for any intermediate value of n. Figure 1 depicts the relation $N_*(z_*)$ corresponding to the first formula of (7) for $n = 1/2$.

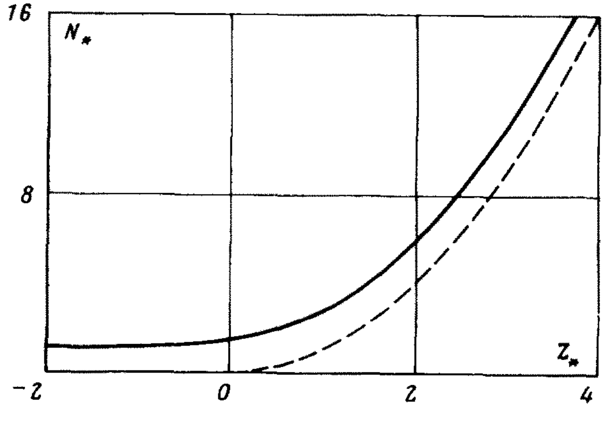

Fig. 1

The dashed line indicates the asymptotic curve of the outer solution constructed at zero counterpressure.

The results obtained give an idea of the perturbation front structure appearing in the presence of counterpressure which is small compared with $\rho_1 U^2$. The values of the hydrodynamic parameters approach asymptotically the given values of the unperturbed flow parameters at one of the boundaries of this structured region having the thickness of the order of ε^n.

REFERENCES

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